On a mathematical model of dynamics of the elastic wedge-shaped medium with radiating defect

Vyacheslav Berkovich[1,*] , *Dmitrii* Pozharskii[2], *Victor* Poltinnikov[2], and *Anna* Morozova[3]

[1]Don Cossacks' State Institute of Food Industry and Business (brunch) «MGUTU» named after K.G.Razumovsky (First Cossacks' University), 55, Semashko, 344007, Rostov-on-Don, Russia
[2]Don State Technical University, 1, Gagarin sq., 344010, Rostov-on-Don, Russia
[3]Rostov State Transport University, 2, Rostovskogo Strelkovogo Polka Narodnogo Opolcheniya sq., 344038, Rostov-on-Don, Russia

Abstract. In the paper the mixed boundary value problem of antiplane vibrations is considered in the elastic wedge-shaped medium containing the radiating defect J_2. Radiating generators are assumed to be located on defect boundaries and on the interval J_1 of the wedge free boundary as well. The problem of reconstructing the wave field in the whole wedge-shaped region with its boundary is stated. A number of problems of analyzing acoustic emission signals by radiating defect are reduced to the problem considered in connections with using non-destructive testing elements of the technological equipment under exploitation. The problem in question is reduced to studying the solvability problems of the equivalent boundary integral equation system both for stress saltus on the defect $J_2$and contact stresses on the interval J_1 of the upper plane of the wedge.

1 Introduction

The aim of the present paper is mathematical modeling of a pre-fracture state of the construction unit representing the junction of angular elastic elements. It is investigated correctness problems of applying mathematical modeling method for the wave process arising in angular elements examined by non-destructive testing methods both in hard industry enterprises and in ones of agricultural machinery. Under long dynamic exploitation of the technological equipment it appears the stress singularity at the angular point. In its neighborhood there arises the defect growing to the angular point (stress concentration) and generating the acoustic radiation (acoustic emission - AE). Non-destructive testing methods are worked out in details in [1-6]. At the paper the pre-fracture state is considered, provided the appearance of the radiating defect takes place only in one of angular elements. The angular element is modeling by elastic body of wedge-shaped medium, one of its planes is stiffly connected with other angular elements, radiating defect is modeling by the linear radial cut of finite length, antiplane vibration generators being located on the cut

* Corresponding author: bvn06@yandex.ru

boundaries. The contact interaction zone is modeling by the harmonic oscillating punch under antiplane deformation (Fig.1).

1. The boundary value problem is formulated for the dynamic elasticity equations in the domain Ω, presenting the angle of span α with cut J_2, simulating the defect located on the segment $[a_2, b_2]$ of the line G_{α_0}. Oscillating coherent generators of the antiplane shear displacements $f_2(r)e^{-i\omega t}$ of the equal intensity are located on the banks of $J_2^{\pm}$.

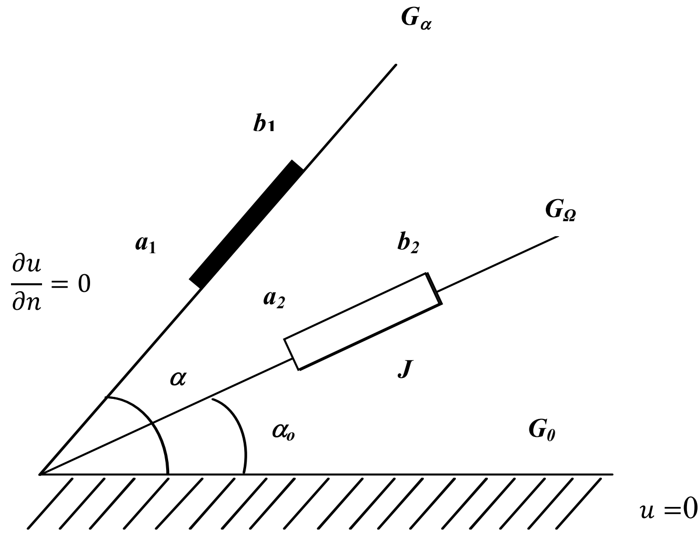

Fig. 1. Elastic wedge-shaped medium with radiating defect.

On the upper plane G_α harmonic oscillation generators $f_1(r)e^{-i\omega t}$ are located on the finite segment $[a_1, b_1]$ as well. The rest of the upper plane G_α is assumed to be unloaded and the lower plane G_0 is stiffly connected. In the vertex $r = 0$ there are no radiation sources and Somerfield's radiation conditions take place at infinity. Under the steady oscillations there states the problem of finding the unknown contact stresses in the zone J_1 of the contact interaction with the punch, stress saltus on the cut J_2 and the reconstructing displacement field in the whole domain Ω including the upper boundary G_α.

Under harmonic oscillations with the frequency ω displacements $U(r, \varphi, t)$ obey the dynamic elasticity equations and are sought in the form $U(r, \varphi, t) = u(r, \varphi)\, exp(-i\omega t)$. The problem in question as it is known in the classic statement is reduced to the next boundary value problem for the Helmholtz equation of complex displacement amplitude $u(r, \varphi)$:

$$\Delta u + K^2 u = 0, K^2 = D\omega^2/\mu \qquad (1)$$

$$u|_{\varphi=\alpha} = f_1(r), r \in (a_1, b_1)$$

$$\sigma_{r\varphi}\big|_{\varphi=\alpha} = 0, r \notin (a_1, b_1) \qquad (2)$$

$$u|_{\varphi=0} = 0, \qquad u|J_2^{\pm} = f_2(x, y), \qquad (x, y) \in J_2^{\pm},$$

where D, μ are density and shear modules of the wedge material, $J_2^{\pm}$ is the left and right cut boundaries of the cut $J_2 = J_2^- \cup J_2^+$ respectively . No radiation sources are assumed to be on the vertex $r = 0$, displacements vanish at infinity and Somerfield's radiation conditions principle take place:

$$\frac{\partial u}{\partial r} - iKu = o\left(r^{-\frac{1}{2}}\right), r = \sqrt{x^2 + y^2} \to \infty \tag{3}$$

The lack of radiation sources on the vertex $r = 0$ provides the fulfilment the Saint-Venant principle and the existence of the solution in Sobolev space $W_2^1(\Omega)$, the norm being given by the traditional way.

Solving the problem stated above is based on its reducing to the equivalent boundary integral equations (BIE) about the unknown (dimensionless) contact stresses amplitude $\mu^{-1}\sigma_{r\varphi}\big|_{\varphi=\alpha} = q_1(r),\ a_1 < r < b_1$ on the upper plane and the unknown amplitude saltus of (dimensionless) stresses $\left[\mu^{-1}\sigma_{r\varphi}\right]\big|_{\varphi=\alpha_0} = q_2(r), a_2 < r < b_2$ on the defect J_2.

The statement of the problem in question leads to the next boundary value problem in the domain Ω. To solve the problem there fulfils the construction of Green function $G(r,\varphi|\rho,\psi)$ in the wedge-shaped domain without defect. Green function obeys the non-homogeneous Helmholtz equation and boundary conditions as follows:

$$\Delta G + K^2 G = -\frac{1}{4\pi r}\delta(\rho - r)\delta(\psi - \varphi), K^2 = D\omega^2/\mu \tag{4}$$

$$\frac{\partial G}{\partial n}\bigg|_{\varphi=0} = G\bigg|_{\varphi=\alpha} = 0\ , \tag{5}$$

where $\delta(x)$ is Dirac's function, n is the external normal to the boundary. Green function method is worked out in details [7-9] when solving static problems. The same method permits to solve boundary value problems of the dynamic elasticity and reconstruct the wave field generated by all vibration sources in the whole wedge-shared medium considered. As in the statement of the main problem (1), (2) no radiation sources are assumed to be on the vertex $r = 0$ and the same conditions at infinity take place.

To construct Green function obeying non-homogeneous Helmholtz equation and boundary conditions (4), (5) the Kontorovich-Lebedev integral transform methods are used in the form

$$\overline{f}(\tau) = \int_0^\infty f(r)K_{-i\tau}(\kappa r)\frac{dr}{r}$$
$$f(r) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \bar{f}(\tau)I_{-i\tau}(\kappa r)\tau d\tau \tag{6}$$

where $I_\nu(\kappa r), K_\nu(\kappa r)$ are modified Bessel functions.

Series of awkward transformations connected with solving the problem (4), (5) results the expression of Green function as follows:

$$G(r,\varphi|\rho,\psi) = \int_0^\infty \overline{G}(\tau,\varphi,\psi)K_{-i\tau}(\kappa r)I_{-i\tau}(\kappa\rho)\,\tau d\tau,$$

$$\overline{G}(\tau,\varphi,\psi) = \begin{cases} 2\dfrac{sh\psi\tau ch(\alpha-\varphi)\tau}{ch\alpha\tau}, \varphi > \psi, \\ 2\dfrac{sh\varphi\tau ch(\alpha-\psi)\tau}{ch\alpha\tau}, \varphi < \psi \end{cases}$$

To obtain the boundary integral equation, to use the well-known integral representation of the regular solution of Helmholtz equation in the next form

$$u(x,y) = -\frac{1}{2\pi}\int_{L_R}\left\{G(x,y|\xi,\eta)\frac{\partial u}{\partial n} - u(\xi,\eta)\frac{\partial}{\partial n}G(x,y|\xi,\eta)\right\}dl_n \quad (7)$$

$$L_R = G_\alpha^R \cup G_0^R \cup J_2^+ \cup J_2^- \cup G_R, \quad J = J_2^+ \cup J_2^-$$

In the formulas (7) contour G_R is the part of the circumference of the finite radius R (with the center in the wedge vertex), closing the angular domain, G_α^R, G_0^R are parts of G_α, G_0 truncated by G_R, n is external normal to the boundary, $J_2^\pm$ are banks of the cut J_2. By virtue of coherence and equal intensity of vibration sources on the cut banks, boundary conditions, vanishing conditions for displacements and Green function as $R \to \infty$, as well as radiation conditions (3) we obtain the limit expression of displacement in the whole wedge-shaped domain Ω:

$$u(x,y) = \frac{1}{2\pi}\int_{J_+} G\left[\frac{\partial u}{\partial n}\right]_{J_2} dl_n - \frac{1}{2\pi}\int_a^b G|_{\psi=\alpha}\frac{\partial u}{\partial n}\bigg|_{J_1} dl_n \quad (8)$$

In (8) let observation point (x,y) tends the left bank of the cut J_2, then let the latter tends to the segment J_1 of the upper wedge plane where vibration sources are given and pass to the polar coordinates (r,φ). There results the boundary integral equation (BIE) system about the amplitude stress saltus q_2 on the cut J_2 and amplitude contact stresses q_1 on the segment J_1. BIE system takes the form:

$$\int_{a_1}^{b_1} k_{11}(r,\rho)q_1(\rho)d\rho + \int_{a_2}^{b_2} k_{12}(r,\rho)q_2(\rho)\,d\rho = f_1(r), \quad a_1 \le r \le b_1 \quad (9)$$

$$\int_{a_1}^{b_1} k_{21}(r,\rho)q_1(\rho)d\rho + \int_{a_2}^{b_2} k_{22}(r,\rho)q_2(\rho)\,d\rho = f_2(r), a_2 \le r \le b_2 \quad (10)$$

Let us introduce matrixes as follows

$$k(r,\rho) = \begin{pmatrix} k_{11}(r,\rho) & k_{12}(r,\rho) \\ k_{21}(r,\rho) & k_{22}(r,\rho) \end{pmatrix},$$

$$k(r,\rho) = \frac{2}{\pi^2}\int_0^\infty K_{-i\tau}(\kappa\rho)K_{-i\tau}(\kappa r)k(\tau)\tau sh\pi\tau d\tau, \kappa = -iK$$

$$k(\tau) = \begin{pmatrix} K_{11}(\tau) & K_{12}(\tau) \\ K_{21}(\tau) & K_{22}(\tau) \end{pmatrix} = \begin{pmatrix} \frac{th\alpha\tau}{\tau} & \frac{sh\alpha_0\tau}{\tau ch\alpha\tau} \\ \frac{sh\alpha_0\tau}{\tau ch\alpha\tau} & \frac{sh\alpha_0\tau ch(\alpha-\alpha_0)\tau}{\tau ch\alpha\tau} \end{pmatrix}.$$

Let us consider the auxiliary BIE, constructed on the base of (9), (10) and written in the abbreviated vector-matrix form $\left(a = \min_{i=1,2} a_i\ , \quad b = \max_{i=1,2} b_i\right)$:

$$\int_a^b k(r,\rho) \bullet q(\rho)d\rho = \tilde{f}(r), a < r < b \quad (11)$$

$$\tilde{f} = \begin{pmatrix} \tilde{f}_1(r) \\ \tilde{f}_2(r) \end{pmatrix},\ q = \begin{pmatrix} q_1(\rho) \\ q_2(\rho) \end{pmatrix}$$

In the correlation (11) $\tilde{f}_{1,2}(r)$ are results of extending of functions $f_{1,2}(r)$ to the interval(a,b), matrix function $k(\tau)$ is real both on the imaginary axis and on the real one where $k(\tau)$ is positively defined.

2. To investigate the solvability problems for the BIE system the next theorem is established.

2 Theorem

Operator K of the left hand side (11) is uniquely inverted as operator acting in vector function spaces:

$$K: H(a,b) \rightarrow W_2^{\frac{1}{2}}(a,b)$$
$$H(a,b) \subset W_2^{-\frac{1}{2}}(a,b)$$

where $W_2^{\gamma}(a,b), \gamma = \pm {}^1\!/_2$ are Sobolev-Slobodetsky spaces of fractional smoothness.

To prove the theorem, to put $\kappa > 0$ temporary. Then by virtue of the positive definition of matrix $k(\tau), \tau \in R^1$ the operator K appears to be positively defined as well and induces the space $H(a,b)$ of generalized solutions of the equation (11), one being introduced by the norm

$$\|q\|_{H(a,b)} = \left(\int_0^\infty Q^T(x) \bullet K(x) \bullet Q^*(x) dx\right)^{\frac{1}{2}}$$

$$Q(x) = \sqrt{xsh\pi x} \int_a^b q(s) K_{-ix}(\kappa s) ds\,, \quad 0 < x < \infty$$

and the scalar product (* means complex conjugation, T is the transposition option):

$$(q_1, q_2)_H = \int_0^\infty Q_1^T(x) \bullet K(x) \bullet Q_2^*(x) dx$$

The use of Riesz's theorem on uniqueness of representation of linear continual functional in the Hilbert space [5] adduces to the solvability condition for BIE system (11) as follows

$$M^2 = \int_0^\infty F^T(x) \bullet K^{-1}(x) \bullet F^*(x) dx < \infty\,,$$

$$F(x) = \sqrt{xsh\pi x} \int_0^\infty \tilde{f}(s) K_{-ix}(\kappa s)\, ds.$$

Constructing the special two-sided estimation for the magnitude M^2 obtaining by the use of the integral representation of McDonald function $K_{-ix}(\kappa s)$ [4] and Parceval equality for Fourier integral transform adduce to the condition $\tilde{f} \in W_2^{\frac{1}{2}}(a,b)$. It is well-known the space $W_2^{-1/2}(a,b)$ is conjugate to the space $W_2^{1/2}(a,b)$ [6]. Then from the Riesz's theorem it points out both the existence of the unique solution $q \in W_2^{-1/2}(a,b)$ for any right hand side $\tilde{f} \in W_2^{\frac{1}{2}}(a,b)$ and the imbedding $H(a,b) \subset W_2^{-1/2}(a,b)$.

The result described is in accordance with known results on boundary properties of functions belonging to Sobolev spaces $W_2^1(a,b)$ in which the solution of boundary value problems is searched by the dynamic elasticity methods.

The passage to the initial case $\kappa = -ik$ is provided by the analytical continuation principle [12] since all functions are analytical with respect to κ in the domain $Re\,\kappa \geq 0$, $\kappa \neq 0$ of the complex plane, where, in part, the point $\kappa = -ik$ is located.

It permits to ascertain the unique solvability of the initial boundary value problem (1), (2) in the Sobolev space $W_2^1(\Omega)$ for the whole wedge-shaped domain and there results the inequality :

$$\|q\|_{H(a,b)} \leq C\|\tilde{f}\|_{W_2^{\frac{1}{2}}(a,b)}, \; C = const$$

meaning the correct solvability of the problem in question permitting to apply varies analytical (for example, methods in[12]) and numerical methods to approach sought-for-functions q_1, q_2 as solutions of the BIE system (9), (10) .

The consequent use of described results to narrowing of functions $q_i, f_i (i = 1,2)$ from the domain (a,b) to the initial ones $(a_i, b_i) \subset (a,b)$ leads to the unique resolution of the initial BIE system (9), (10) in spaces of the fractional smoothness. It means there exists the unique solution $q_i \in W_2^{\frac{-1}{2}}(a_i, b_i)$ for any right hand side function

$$f_i \in W_2^{\frac{1}{2}}(a_i, b_i), \; i=1,2.$$

The reconstruction of displacement wave field in Ω and in the boundary G_α may be fulfilled by representation (8) which is presented by means of displacements' amplitude $f_2(x,y)$ when radiating AE from defect boundaries. The displacement wave field may be considered as the base to the statement of the inverse problem of reconstructing the displacements' amplitude $f_2(r)$ on the defect J_2 by means of direct displacement measurements. It may be done when constructing the displacements' amplitude frequency reply on the unloaded part of the boundary G_α and the sequel application of the least square method [14,15].

References

1. S.I. Builo, *Physical, mechanical Statistical and chemical aspects of acoustic emission diagnostics* (Southern Federal University Press, Rostov-on-Don, 2017)
2. V.I. Ivanov, V.A. Barat, *Acoustic emission diagnostics* (Publishing «Spectr», Moscow, 2017)
3. S.I. Builo, Russ. J. Nondestr. Test. **54(4)**, 237-242 (2018)
4. S.I. Builo, Russian Journal of Nondestuctive Testing **11**, 32-45 (2012)
5. S.I. Builo, *Physics and Mechanics of New Materials and Their Applications* N(ova Science Publishers, New York, 2013)
6. S.I. Builo, D.M. Kuznetsov, V.L. Gaponov, Russian Journal of Nondestructive Testing **55(11)**, 803–807 (2019)
7. M.Yu. Kashtalyan, J.J. Rushchitsry, Int. Appl. Mech. **46(1)**, 1-17 (2010)
8. D.A. Pozharskii, J. Appl. Math. and Mech. **80(4)**, 489-495 (2018)
9. B. Sobol, A. Krasnoschekov, A. Soloviev, ZAMM **95(11)**, 1302-1314 (2015)
10. E. Pan, W. Chen, *Static Green's function in anisotropic media* (Cambridge University

Press, New York, 2015)

11. Y. Xu, R.L. Jackson, J. Tribology **140(1)**, 1401-1404 (2018)
12. V.N. Berkovich, A.V. Babkin, Int. Res. J. **58(23)**, 101-104 (2017) DOI: http//doi.org/10.23670/IRJ. 2017.58.042.
13. V.N. Berkovich, Int. Res. J. **2(92)**, 19-24 (2020) DOI: http//doi.org/10.23670/IRJ.2020.92.2.003
14. R. Nedin, A. Vatulyan, Int. J. Solids Struct. **50**, 2107–2114 (2013)
15. V.N. Berkovich, S.I. Builo, Russian Journal of Nondestuctive Testing **4**, 262-267 (2019)