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About one boundary-value problem arising in modeling dynamics of groundwater

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Abstract. Modeling the movement of moisture in the soil is of great importance for assessing the impact of agricultural land on surface water bodies and, consequently, on the natural environment and humans. This is because huge volumes of pollutants from the fields (pesticides, mineral fertilizers, nitrates, and nutrients contained in them) are transferred to reservoirs by filtering moisture. Different methods solve all these tasks. The method of natural analogies is based on the analysis of graphs of fluctuations in groundwater level. To apply this method on irrigated lands, it is necessary to have a sufficiently studied irrigated area with similar natural, organizational and economic conditions. The successful application of this method, based on the fundamental theory of physical similarity, mainly depends on the availability of a sufficiently close comparison object, which is quite rare in practice. Physical modeling is often used to construct dams and other hydraulic structures. Previously, the method of electrical modeling was also widely used. It was further found that nonlocal boundary conditions arise in the problems of predicting soil moisture, modeling fluid filtration in porous media, mathematical modeling of laser radiation processes, and plasma physics problems, as well as mathematical biology.

1 Introduction

At present, boundary value problems for equations of mixed type have become an important part of the modern theory of partial differential equations. One of the main problems in the theory of partial differential equations is the study of mixed-type equations, which is of theoretical and practical interest. In 1959, I.N. Vekua pointed out the importance of the problem of equations of mixed type in connection with problems in the theory of infinitesimal bendings of surfaces. The problem of the outflow of a supersonic jet from a vessel with flat walls is reduced to the Tricomi problem for the Chaplygin equation (a degenerate equation of mixed type). There are several works by F. Tricomi, S. Gelderstedt, A. V. Bitsadze, M. S. Salakhitdinov, T.D. Dzhuraev and their students in which the main mixed boundary value problems are studied, and new correct problems are posed for the equations of the elliptic-hyperbolic, parabolic-hyperbolic types of the first kind, i.e., equations for which the degeneracy line is not a characteristic.

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In recent years, a large number of papers have appeared devoted to the study of equations of composite and mixed-composite types. Correct boundary value problems for equations of mixed-composite type, the main part of which contains an elliptic-hyperbolic operator, were first formulated by A.V. Bitsadze (see [1], [2]). These problems and some of their generalizations have now been studied in detail.

We note that the results of all the above works were obtained for equations of the first kind, and for equations of the second kind of the third order, boundary value problems have not been previously studied.

Therefore, the study of boundary value problems for mixed-type equations of the second kind seems very relevant and little studied. We note the works [3-6].

In this paper, we study a local boundary value problem for equations of mixed composite type of the second kind, i.e., for an equation where the line of degeneracy is a characteristic.

2 Statement of the problem

Consider the equation

$$\frac{\partial}{\partial y}(Lu)=0, \tag{1}$$

in the domain of $D=D_1\cup D_2\cup OB$, and the domain D_2 limited at $x<0$ characteristics

$$OC:y-\frac{2}{m+2}(-x)^{\frac{m+2}{2}}=0,\ BC:y+\frac{2}{m+2}(-x)^{\frac{m+2}{2}}=1,\ OB:x=0$$

Equations

$$Lu\equiv\frac{1+\operatorname{sgn}x}{2}\left(\frac{\partial^2u}{\partial x^2}-\frac{\partial u}{\partial y}\right)+\frac{1-\operatorname{sgn}x}{2}\left(\frac{\partial^2u}{\partial x^2}-(-x)^m\frac{\partial^2u}{\partial y^2}\right)=0 \tag{2}$$
$$m\in\left(-\frac{16}{9};-\frac{7}{4}\right)$$

and the domain D_1 at $x>0$ limited by segments OA, AD, BD, OB of straight lines $y=0,\ x=1,\ y=1,\ x=0$, respectively.

The general solution of equation (1) can be represented as [5]:

$$u(x,y)=z(x,y)+\omega(x) \tag{3}$$

where $z(x,y)$ is regular solution of equation (2) in the domain D_1, and in the domain D_2 is a generalized solution of the class R. Denote $\omega(x)$ in the following form:

$$\omega(x)=\begin{cases}\omega_1(x) & at \quad x>0\\ \omega_2(x) & at \quad x<0\end{cases}$$

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and $\omega_1(x)$ has all the derivatives in equation (1), and the smoothness of the function is given by the definition of a generalized solution of the class R of equation (1).

Dirichlet problem. Required to define a function $u(x,y)$ that has the following properties:

a) $u(x,y)\in C(\overline{D})$;

b) function $u(x,y)$ is a generalized solution of equation (1) of class R in the domain D_2, and in the domain D_1 is regular;

c) the gluing condition is satisfied on the degeneracy line

$$-\lim_{x\to -0}\frac{\partial u}{\partial x}=\lim_{x\to +0}\frac{\partial u}{\partial x};$$

d) u_x continuous up to the transition line both on the left and on the right;

e) satisfies the boundary conditions

$$u|_{OA}=\tau_1(x),\ u|_{AD}=\psi(y),\ u|_{BD}=\psi_1(x),$$
$$[u-w(x)]|_{OC}=\psi_2(x),\ u|_{BC}=\psi_3(x),$$

where $\tau_1(x),\psi(x),\psi_1(y),\psi_2(x),\psi_3(x)$ are given sufficiently smooth functions and $\tau_1(0)=\psi_2(0),\ \tau_1(1)=\psi(0),\ \psi(1)=\psi_1(1),\ \psi_1(0)=\psi_3(0)$.

$$\psi_2\left(-\left(\frac{m+2}{4}\right)^{\frac{2}{m+2}}\right)=\psi_3\left(-\left(\frac{m+2}{4}\right)^{\frac{2}{m+2}}\right)$$

here $-\left(\frac{m+2}{4}\right)^{\frac{2}{m+2}}$ is coordinate of point C to x.

Note that this problem is in the case $m=0$ studied in [2] and in the case $-1<m<0$ considered in [1].

Without loss of generality, we can assume that $w(0)=0,\ w(1)=1$. Based on (3) and boundary conditions, the Dirichlet problem is reduced to the definition of a regular solution in the domain D_1, a generalized solution of the class R in the domain D_2 equation (2) satisfying the conditions

$$z|_{OA}=\tau_1(x)-w_1(x),\quad z|_{AD}=\psi(y),\quad z|_{BD}=\psi_1(x)-w_1(x),$$
$$z|_{OC}=\psi_2(x),\qquad z|_{BC}=\psi_3(x)-w_2(x).$$

3 Uniqueness of solutions to the problem

We will prove the uniqueness of the problem under consideration by the method of energy integrals. In the domain of D_1 we have the equation $z_{xx}-z_y=0$

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$$\iint_{D_1} z(z_{xx} - z_y)dxdy = \iint_{D_1} (zz_{xx} - zz_{yy})dxdy = 0.$$

Can express zz_{xx} via $zz_{xx} = \frac{\partial}{\partial x}(zz_x) - z_x^2$ then the last equality takes the form:

$$\iint_{D_1} \frac{\partial}{\partial x}(zz_{xx}) - z_x^2 - zz_y)dxdy = 0.$$

Applying Green's formula, we get the following:

$$-\int_0^1 zz_x dy - \iint_{D_1} z_x^2 dxdy = \iint_{D_1} zz_y dxdy$$

Let us show that the second integral of the left side of the equality is equal to zero. To do this, we use Green's formula, and since

$$-\int_0^1 zz_x dy - \iint_{D_1} z_x^2 dxdy = 0. \quad (4)$$

we have

$$\int_0^1 z(0,y)z_x(0,y)dy \le 0. \quad (5)$$

Integrating the identity

$$z\left[z_{xx} - (-x)^m z_{yy}\right] = \frac{\partial}{\partial x}(zz_{xx}) - \frac{\partial}{\partial y}\left[(-x)^m zz_y\right] - z_x^2 + (-x)^m z_y^2$$

by domain D_2 and applying Green's formula to the right side of equality, we have

$$\int_{\partial D_2} z\left[z_x dy + (-x)^m z_y dx\right] - \iint_{D_2} \left[z_x^2 + (-x)^m z_y^2\right]dxdy = 0$$

Let us divide the first integral into three parts, i.e., integrating by parts, respectively; we have

$$\int_C^B z\left[z_x dy + (-x)^m z_y dx\right] + \int_C^O z\left[z_x dy + (-x)^m z_y dx\right] \ge 0.$$

Consequently

$$\int_0^1 z(0,y)z_x(0,y)dy \geq 0. \qquad (6)$$

Then, inequalities (5) and (6) lead to the equality

$$\int_0^1 z(0,y)z_x(0,y)dy = 0.$$

Therefore, from (4), we obtain

$$\iint_{D_1} z_x^2 dxdy = 0,$$

means, $z(x,y)=\mu(y)$, from the boundary condition $z\big|_{AD}=0$ follows $z(x,y)\equiv 0$ in D_1, and from $z\big|_{BD}=-\omega_1(x)$ we get $u(x,y)\equiv 0$ in D_1. Insofar as $z\big|_{OC}=0$, $z\big|_{OB}=0$ and from the uniqueness of the Cauchy problem in the hyperbolic domain we obtain $u(x,y)\equiv 0$ in D_2, which was to be proved.

4 Existence of a solution to the problem

It is known that the solution of the Cauchy problem for the equation $L_2 z=0$ in the domain of D_2 has the form

$$z(\xi,\eta)=\int_0^{\xi}(\eta-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}T(\zeta)d\zeta+\int_{\xi}^{\eta}(\eta-\zeta)^{-\beta}(\zeta-\xi)^{-\beta}N(\zeta)d\zeta \qquad (7)$$

where

$$N(\zeta)=\frac{1}{2\cos\pi\beta}T(\zeta)-\gamma_2\nu(\zeta), \qquad (8)$$

$$\tau(y)=z(0,y), \quad 0\leq y\leq 1,$$

$$\nu(y)=\lim_{x\to -0}\frac{\partial u}{\partial x}=[2(1-2\beta)]^{-2\beta}\lim_{\eta-\xi\to -0}(\eta-\xi)^{2\beta}\left(\frac{\partial u}{\partial\xi}-\frac{\partial u}{\partial\eta}\right),$$

Because $z(x,y)$ is the generalized solution of the Cauchy problem for the equation $L_2 z=0$ in the domain of D_2 from the class R_2 then has representation (7) and

$$\tau(y)=\tau(0)+\int_0^{y}(y-t)^{-2\beta}T(t)dt, \qquad (9)$$

and functions $T(t)$ and $\nu(t)$ are continuous and integrable on $(0,1)$, where

$$\gamma_2 = [2(1-2\beta)]^{2\beta-1} \frac{\Gamma(2-2\beta)}{\Gamma^2(1-\beta)},\ \beta = \frac{m}{2(m+2)}.$$

To represent the solution of the equation $L_2 z = 0$ in the domain of D_2 satisfying the boundary conditions $z|_{OB} = \tau(y),\ z|_{AD} = \psi(y),\ z|_{OA} = \tau_1(x) - w_1(x)$, we use the solution of the first boundary value problem, i.e.,

$$z(x,y) = \int_0^y \tau(\eta) G_\xi(x,y;0,\eta) d\eta + \\ + \int_0^1 [\tau_1(\xi) - w_1(\xi)] G(x,y;\xi,0) d\xi - \int_0^y \psi(\eta) G_\xi(x,y;1,\eta) d\eta, \tag{10}$$

where $G(x,y;\xi,\eta)$ Green's function of the first boundary value problem for the heat equation has the form[7-8]:

$$G(x,y;\xi,\eta) = \sum_{n=-\infty}^{+\infty} [z(x,y;\xi+2n) - z(x,y;-\xi+2n,\eta)]$$

and

$$z(x,y;\xi,\eta) = \frac{1}{2\sqrt{\pi}} \begin{cases} \frac{1}{\sqrt{y-\eta}} e^{-\frac{(x-\xi)^2}{4(y-\eta)}} & at\ \ y > \eta \\ 0 & at\ \ y \le \eta \end{cases}$$

To define an unknown function $w_1(x)$ implement the condition

$$z|_{BD} = \psi_1(x) - w_1(x) \tag{11}$$

Based on (10), (11) and that in $BD: y = 1$ then we get

$$\psi_1 - w_1(x) = \int_0^1 \tau(\eta) G_\xi(x,1;0,\eta) d\eta + \int_0^1 \tau_1(\xi) G(x,1;\xi,0) d\xi - \\ - \int_0^1 w_1(\xi) G(x,1;\xi,0) d\xi - \int_0^1 \psi(\eta) G_\xi(x,1;1,\eta) d\xi$$

The last equality can be expressed as follows

$$w_1(x) - \int_0^1 w_1(\xi) G(x,1;\xi,0) d\xi = g(x), \tag{12}$$

where

$$g(x) = \psi_1(x) - \int_0^1 \tau(\eta) G_\xi(x,1;0,\eta) d\eta - \\ - \int_0^1 \tau(\xi) G(x,1;\xi,0) d\xi + \int_0^1 \psi(\eta) G_\xi(x,1;1,\eta) d\eta.$$

Equation (12) is an integral Fredholm equation of the second kind, the solvability of which follows from the uniqueness of the solution to the problem and is determined by the formula

$$w_1(x)=g(x)+\int_0^1 g(\xi)R(x,\xi;-1)d\xi$$

Calculating the derivative $\frac{\partial z}{\partial x}$, then letting x tend to zero, taking into account (9) and the Dirichlet transformation, we have

$$\begin{aligned}\nu(y)=&\frac{2\beta}{\sqrt{\pi}}\int_0^y T(t)dt\int_t^y (y-\eta)^{-\frac{1}{2}}(\eta-t)^{-2\beta-1}d\eta+\int_0^y T(t)dt\int_t^y K_1(y,\eta)(\eta-t)^{-2\beta}d\eta+\\&+\int_0^1 T(s)ds\int_0^1 G_x(0,y;t,0)dt\int_s^1(\eta-s)^{-2\beta}G_\xi(t,1;0,\eta)d\eta+\\&+\int_0^1 T(s)ds\int_0^1 G_x(0,y;z,0)dz\int_0^1 R(z,t;-1)dt\int_s^1 G_\xi(t,1;0,\eta)(\eta-s)^{-2\beta}d\eta+\Phi_2(y).\end{aligned}\tag{13}$$

where

$$K_1(y,\eta)=G_{\xi x}(x,y;0,\eta)\Big|_{x=0}=\frac{1}{2\sqrt{\pi}}\left[\frac{1}{(y-\eta)^{\frac{3}{2}}}\sum_{\substack{n=-\infty\\n\neq0}}^{+\infty}e^{\frac{n^2}{y-\eta}}-\frac{1}{(y-\eta)^{\frac{5}{2}}}\sum_{\substack{n=-\infty\\n\neq0}}^{+\infty}n^2e^{-\frac{n^2}{y-\eta}}\right],$$

$$\Phi_2(y)=-\int_0^1\Phi_1(\xi)G_x(0,y;\xi,0)d\xi+\int_0^1\tau_1(\xi)G_x(0,y;\xi,0)d\xi-$$

$$-\int_0^y\psi(\eta)G_{\xi x}(0,y;1,\eta)d\eta.$$

We extend the first and second integrals on the right side of (13) concerning $\boldsymbol{t}$ to $(0,1)$ those.

$$\int_0^1 K_2(y,t)T(t)dt=\frac{2\beta}{\sqrt{\pi}}\int_0^y T(t)dt\int_0^y(y-\eta)^{-\frac{1}{2}}(\eta-t)^{-2\beta-1}d\eta,$$

$$\int_0^1 K_3(y,t)T(t)dt\equiv\int_0^y T(t)dt\int_t^y K_1(y,\eta)(\eta-t)^{-2\beta}d\eta,$$

Where

$$K_2(y,t)=\begin{cases}\frac{2\beta}{\sqrt{\pi}}\int_t^y(y-\eta)^{-\frac{1}{2}}(\eta-t)^{-2\beta-1}d\eta & at\ \ 0\le t\le y,\\ 0 & at\ \ y<t\le1,\end{cases}$$

$$K_3(y,t)=\begin{cases}\int\limits_t^y (y-t)^{-2\beta} K_1(y,\eta)\,d\eta & at\ \ 0\le t\le y,\\ 0 & at\ \ y<t\le 1.\end{cases}$$

Then (13) has the form

$$\nu(y)=\int\limits_0^1 K(y,t)T(t)dt+\Phi_2(y), \tag{14}$$

where

$$K(y,t)=K_2(y,t)+K_3(y,t)+\int\limits_0^1 G_x(0,y;t,0)dt\cdot\int\limits_s^1(\eta-s)^{-2\beta}G_\xi(t,1;0,\eta)d\eta+$$
$$+\int\limits_0^1 G_x(o,y;z,0)dz\int\limits_0^1 R(z,t;-1)dt\int\limits_s^1 G_\xi(t,1;0,\eta)(\eta-s)^{-2\beta}d\eta.$$

From (8), we find

$$-\nu(y)=\frac{1}{\gamma_2}\left[N(y)-\frac{1}{2\cos\pi\beta}T(y)\right]. \tag{15}$$

Taking into account the gluing condition and excluding $z_x(0,y)=\nu(y)$ from (14) and (15), we have

$$N(y)-\frac{1}{2\cos\pi\beta}T(y)=\gamma_2\int\limits_0^1 K(y,t)T(t)dt+\gamma_2\Phi_2(y)$$

or

$$T(y)+2\gamma_2\cos\pi\beta\int\limits_0^1 K(y,t)T(t)dt=2\cos\pi\beta N(y)-2\gamma_2\cos\pi\beta\,\Phi_2(y). \tag{16}$$

The study of equation (16) shows that it is an integral Fredholm equation of the second kind with a weak singularity. Its unique solvability follows from the uniqueness of the solution to the problem. Solutions of the integral equation (16) can be written using the resolvent as

$$T(y)=2\cos\pi\beta[N(y)-\gamma_2\Phi_2(y)]-4\gamma_2\cos^2\pi\beta\times$$
$$\times\int\limits_0^1 R_1(y,s;\lambda)[N(s)-\gamma_2\Phi_2(s)]ds, \tag{17}$$

where $R_1(y,s;\lambda)$ is the resolvent of equation (16).

Subordinating (7) to the conditions on the characteristics of OC, BC $z|_{OC}=\psi_2(x),\quad z|_{BC}=\psi_3(x)-w_2(x)$. and taking into account (17), i.e., on the $OC:\xi=0$ from (7) and denoting $x=-[2(1-2\beta)]^{2\beta-1}\eta^{1-2\beta}$ we get

$$\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\eta^{1-2\beta}\right\}=\int_0^\eta(\eta-\zeta)^{-\beta}\zeta^{-\beta}N(\zeta)d\zeta \qquad (18)$$

Based $BC:\eta=1$ and $x=-[2(1-2\beta)]^{2\beta-1}(1-\xi)^{1-2\beta}$ we get

$$w_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}(1-\xi)^{1-2\beta}\right\}+\int_0^1 K_4(\xi,\zeta)(1-\zeta)^{-\beta}|\xi-\zeta|^{-\beta}d\zeta=F(\xi), \qquad (19)$$

where

$$K_4(\xi,\zeta)=\begin{cases}2\cos\pi\beta N(\zeta)-4\gamma_2\cos^2\pi\beta\int_0^1 N(s)R_1(\zeta,s;\lambda)ds & at\ 0\le\zeta\le\xi,\\ N(\zeta) & at\ \xi<\zeta\le 1\end{cases}$$

$$F(\xi)=\psi_3\left\{-\left[2(1-2\beta)\right]^{2\beta-1}(1-\xi)^{1-2\beta}\right\}-4\gamma_2^2\cos^2\pi\beta\int_0^\xi(1-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}\times$$
$$\times\int_0^1\Phi_2(s)R_1(\zeta,s;\lambda)dsd\zeta+2\gamma_2\cos\pi\beta\int_0^\xi(1-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}\Phi_2(\zeta)d\zeta.$$

The latter system has a solution, which proves the existence of a solution to the Dirichlet problem.

5 Studies on the smoothness of given functions

It can be seen that if we use from (18), we can find $T(y)$, using the fractional operator, we rewrite $N(\eta)$ in the following form:

$$N(\eta)=\frac{\eta^\beta}{\Gamma(1-\beta)}D_{0\eta}^{1-\beta}\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\eta^{1-2\beta}\right\}$$

Therefore, from (19) $w_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}(1-\xi)^{1-2\beta}\right\}$ takes the form:

$$\begin{aligned}&w_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}(1-\xi)^{1-2\beta}\right\}=\psi_3\left\{-\left[2(1-2\beta)\right]^{2\beta-1}(1-\xi)^{1-2\beta}\right\}-\\&-\frac{2\cos\pi\beta}{\Gamma(1-\beta)}\int_0^\xi(1-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}\zeta^\beta D_{0\zeta}^{1-\beta}\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\zeta^{1-2\beta}\right\}d\zeta-\\&-\frac{1}{\Gamma(1-\beta)}\int_\xi^1(1-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}\zeta^\beta D_{0\zeta}^{1-\beta}\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\zeta^{1-2\beta}\right\}d\zeta-\\&-4\gamma_2^2\cos^2\pi\beta\cdot J_1+2\gamma_2\cos\pi\beta\cdot J_2+\frac{4\gamma_2^2\cos^2\pi\beta}{\Gamma(1-\beta)}\cdot J_3\end{aligned} \qquad (20)$$

where

$$J_1=\int_0^\xi(1-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}\int_0^1\Phi_2(s)R_1(\zeta,s;\lambda)dsd\zeta \qquad (21)$$

$$J_2 = \int_0^{\xi} (1-\zeta)^{-\beta} (\xi-\zeta)^{-\beta} \Phi_2(\zeta) d\zeta$$

$$J_3 = \int_0^{\xi} (1-\zeta)^{-\beta} (\xi-\zeta)^{-\beta} \int_0^1 R_1(\zeta,s;\lambda) s^{\beta} D_{0s}^{1-\beta} \psi_2 \left\{ -\left[2(1-2\beta)\right]^{2\beta-1} s^{1-2\beta} \right\} ds d\zeta \quad (22)$$

$$\Phi_2(y) = -\int_0^1 [\psi_1(\xi) - \int_0^1 \tau_1(s) G(\xi,1;s,0) ds + \int_0^1 \psi(\eta) G_{\xi}(\xi,1;1,\eta) d\eta +$$
$$+ \int_0^1 \psi_1(s) R(\xi,s;-1) ds - \int_0^1\int_0^1 \tau_1(t) G(\xi,1;t,0) dt R(\xi,s;-1) ds +$$
$$+ \int_0^1\int_0^1 \psi(\eta) G_{\xi}(t,1;1,\eta) d\eta R(\xi,t;-1) dt] G_x(0,y;\xi,0) d\xi +$$
$$+ \int_0^1 \tau_1(\xi) G_x(0,y;\xi,0) d\xi - \int_0^y \psi(\eta) G_{\xi x}(0,y;1,\eta) d\eta.$$

Let us present some auxiliary expansions of the Green's function involved inside the integral as a kernel

$$G_{\xi}(\xi,1;1,\eta) = \sum_{n=-\infty}^{+\infty} \left[\frac{(\xi-1-2n)}{(1-\eta)^{\frac{3}{2}}} e^{-\frac{(\xi-1-2n)^2}{4(1-\eta)}} - \frac{(\xi+1-2n)}{2(1-\eta)^{\frac{3}{2}}} e^{-\frac{(\xi+1-2n)^2}{4(1-\eta)}} \right],$$

$$G(\xi,1;t,0) = \sum_{n=-\infty}^{+\infty} \left[e^{-\frac{(\xi-t-2n)^2}{4}} - e^{-\frac{(\xi+t-2n)^2}{4}} \right],$$

$$G_x(0,y;\xi,0) = \sum_{n=-\infty}^{+\infty} \left[\frac{\xi+2n}{2y^{\frac{3}{2}}} e^{-\frac{(\xi+2n)^2}{4y}} - \frac{\xi-2n}{2y^{\frac{3}{2}}} e^{-\frac{(\xi-2n)^2}{4y}} \right],$$

$$G_{\xi x}(0,y;1,\eta) = \sum_{n=-\infty}^{+\infty} \left[\frac{(-1-2n)^2}{4(y-\eta)^{\frac{5}{2}}} e^{-\frac{(-1-2n)^2}{4(y-\eta)}} - \frac{(1-2n)^2}{4(y-\eta)^{\frac{5}{2}}} e^{-\frac{(1-2n)^2}{4(y-\eta)}} \right]$$

For (21) to take place, it is necessary Φ_2 was a continuous function, then from the representation $\Phi_2(y)$ it easily follows that ψ_1, τ_1 continuous. Hence from $\int_0^1 \psi(\eta) G_{\xi}(\xi,1;1,\eta) d\eta$ function ψ should look like $\psi(\eta) = (1-\eta)^{\frac{3}{2}} \psi^*(\eta)$ where ψ^* is a continuous function. Now from (22), we will study the function ψ_2. From the definition of integro-differential operators of fractional order $\alpha > 0$ those. From $D_{ax}^{\alpha} f(x) = \frac{d^n}{dx^n} \{D_{ax}^{-(n-\alpha)} f(x)\}$ where $n-1<\alpha<n,\ n\geq 1$, $f(x)\in L(a;b)$ and because $m \in (-\frac{16}{9};-\frac{7}{4})$, $\beta \in (-4;-3{,}5)$, follows that $4 < 1-\beta < 5 \Rightarrow n = 5$ and $D_{0s}^{1-\beta} \psi_2 \{-[2(1-2\beta)]^{2\beta-1} s^{1-2\beta}\} = \frac{d^5}{ds^5} \left[D_{as}^{-(4+\beta)} \psi_2 \{-[2(1-2\beta)]^{2\beta-1} s^{1-2\beta}\} \right]$

$$\frac{d^5}{ds^5}\left[D_{as}^{-(4+\beta)}\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}s^{1-2\beta}\right\}\right]=$$
$$=\frac{d^5}{dx^5}\left[\frac{1}{\Gamma(1-\beta)}\int_0^s(s-\eta)^{3+\beta}\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\eta^{1-2\beta}\right\}d\eta\right] \quad (23)$$

It can be seen that the right side of equation (23) has a weak feature. Therefore, we cannot immediately differentiate it. To avoid this, we will first integrate by parts and then differentiate. Repeating this process five times and putting the result obtained in (22), we choose among them the term with the largest singularity, i.e., $\int_0^s (s-\eta)^{3+\beta}s^{\beta}\psi_2^{(5)}\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\eta^{1-2\beta}\right\}d\eta$. The study of this expression shows that the existence of the integral depends on the continuity of the function belonging to the kernel. To do this, we will do the following: $\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\eta^{1-2\beta}\right\}=\eta^{3-2\beta}\psi_2^*\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\eta^{1-2\beta}\right\}$

Insofar as $x=-[2(1-2\beta)]^{2\beta-1}\eta^{1-2\beta}\Rightarrow\eta=kx^{\frac{1}{1-2\beta}}$ where $k=\frac{(-1)^{\frac{1}{1-2\beta}}}{2(1-2\beta)}$ then we get

$\psi_2(x)=k^{3-2\beta}x^{\frac{3-2\beta}{1-2\beta}}\psi_2^*(x)$. From (20), it can be seen that the function $\psi_3(x)\in C^2(\overline{D}_2)$.

Based on the above results, we will formulate the following theorem:

Theorem If $\psi_3(x)\in C^2(\overline{D}_2)$, ψ_1, τ_1 continuous functions and $\psi(x)$, $\psi_2(x)$ represent in the form $\psi(x)=(1-x)^{\frac{3}{2}}\psi^*(x)$ and $\psi_2(x)=k^{3-2\beta}x^{\frac{3-2\beta}{1-2\beta}}\psi_2^*(x)$, where ψ^*,ψ_2^* continuous functions, then the solution of the Dirichlet problem exists and is unique.

6 Conclusions

Thus, with the help of energy integrals, the uniqueness of the solution of the boundary value problem for the homogeneous equation of parabolic - hyperbolic type of the third order of the second kind is proved. Necessary and sufficient conditions for the existence of a generalized solution to the formulated problem are found. An explicit representation of the solution to the problem under study is obtained. The results obtained and the developed method makes it possible to further investigate similar boundary value problems for a homogeneous parabolic-hyperbolic type equation of the third order of the second kind.

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