E3S Web of Conferences **461**, 01012 (2023) https://doi.org/10.1051/e3sconf/202346101012
RSES 2023

Synthesis of control and observation laws power system based on generalized formula Ackermann for MIMO systems

Irek Galiaskarov[1], *Misrikhan* Misrikhanov[2], *Vladimir* Ryabchenko[3*], *Eugene* Tregubov[3], and *Andrey* Shuntov[2]

[1] JSC «CEMC UES», 4, Belovezhskaya st., Moscow, Russia
[2] National research university «MPEI», 17, Krasnokazarmennaya st., Moscow, Russia
[3] JSC «R&D Center FGC UES», 22/3, Kashirskoe sh., Moscow, Russia

Abstract. For an electric power system (EPS), as a dynamic system with many inputs and many outputs (Multi Inputs Multi Outputs System – MIMO), compact analytical formulas are obtained for calculating the coefficients of the controller matrix and the observer matrix of the state of the solution of the synthesis problem, providing a given placement of eigenvalues along full state vector. These formulas are generalizations to MIMO systems of the well-known Ackermann formula used to design the control of systems with one input and many outputs (Single Input Multi Outputs System – SIMO). The approach is based on the transformations used in the original multi-step (multilevel) decomposition method, as well as a nondegenerate similarity transformation in the form of the Kalman controllability matrix. The obtained formulas are applicable to dynamic systems, for which the dimension of the state space is a multiple of the dimension of the inputs (controls). This limitation is removed by using the Yokoyama transform. These formulas differ in terms of parameterization of the set of equivalent laws. An example of the synthesis of a control law for a synchronous generator in a complex EPS is considered in order to preserve the existing modes of electromechanical oscillations and meet additional requirements (roughness with respect to disturbances and/or increased sensitivity to changes in controlled parameters in a given region or frequency band).

Introduction and problem statement

One of the most well-known explicit calculation formulas used for the synthesis of controllers and observers of linear stationary dynamic systems in the state space with one input and one output, including electric power systems (EPS), are the Ackermann and Bass-Gura formulas [0-4].

Let a fully controlled stationary linear dynamic SIMO-system (Single Input Multi Outputs System) be given

$$\sigma\mathbf{x} = \boldsymbol{A}\mathbf{x} + \mathbf{b}u, \quad (1)$$

where
$\mathbf{x} \in \mathbb{R}^n$ is the state vector, is the scalar input, $u \in \mathbb{R}$, $\mathbb{R}$ – the set of real numbers, $\sigma\mathbf{x}(t) \triangleq \dot{\mathbf{x}}(t)$ for the case of continuous time and discrete time $\sigma\mathbf{x}(t) \triangleq \mathbf{x}(t+1)$.

The condition for complete controllability of the system (1) or complete controllability of the pair corresponds to the nonsingularity of the Kalman controllability matrix $(\mathbf{A}, \mathbf{b})$

$$\Omega(\mathbf{A}, \mathbf{b}) = (\mathbf{b} \quad \boldsymbol{A}\mathbf{b} \quad \cdots \quad \boldsymbol{A}^{n-1}\mathbf{b}) \in \mathbb{R}^{n\times n}. \quad (2)$$

Let also the characteristic polynomial of the matrix $\boldsymbol{A}$ equals

$$det(\lambda\boldsymbol{I}_n - \boldsymbol{A}) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0, \quad (3)$$

where

$\boldsymbol{I}_n$ is the identity matrix of order n, $\alpha_i \in \mathbb{R}$ are the coefficients of the characteristic polynomial, λ is the set of complex numbers $\mathbb{C}$.

Using the feedback law on state variables

$$u = -\mathbf{k}^T\mathbf{x} \quad (4)$$

required to provide a closed system
$\dot{\mathbf{x}} = (\boldsymbol{A} - \mathbf{b}\mathbf{k}^T)\mathbf{x}$,
the following characteristic polynomial:

$$det(\lambda\mathbf{I}_n - \boldsymbol{A} + \mathbf{b}\mathbf{k}^{\mathrm{T}}) = \lambda^n + \beta_{n-1}\lambda^{n-1} + \cdots + \beta_0. \quad (5)$$

If we introduce the notation for the matrix polynomial

$$\Xi(\mathbf{A}, \beta) = \boldsymbol{A}^n + \beta_{n-1}\boldsymbol{A}^{n-1} + \cdots + \beta_1\boldsymbol{A} + \beta_0\boldsymbol{I}_n, \quad (6)$$

then the calculation formula of Ackermann, which determines the vector of coefficients of the controller (dually, the observer of the state) in $\mathbf{k}^T$ (4) looks like [0- 4]

$$\mathbf{k}^T = (0 \quad \cdots \quad 0 \quad 1) \cdot (\mathbf{b} \quad \boldsymbol{A}\mathbf{b} \quad \cdots \quad \boldsymbol{A}^{n-1}\mathbf{b})^{-1} (\boldsymbol{A}^n + \beta_{n-1}\boldsymbol{A}^{n-1} + \cdots + \beta_1\boldsymbol{A} + \beta_0\boldsymbol{I}_n)$$

or taking into account(2), (6)

$$\mathbf{k}^T = (0 \quad \cdots \quad 0 \quad 1) \cdot \Omega^{-1}(\boldsymbol{A}, \mathbf{b}) \cdot \Xi(\boldsymbol{A}, \beta). \quad (7)$$

The main advantage of the Ackermann calculation formula (7), is its explicit form, which allows using the known parameters of the system (1), its characteristic polynomial (3) and a given arrangement of poles (polynomial roots), expressed in coefficients of the charac-

* Corresponding author: ryabchenko.vn@yandex.ru

teristic polynomial (5), calculate the feedback that provides the closed system with the required location of the poles (setting the roots). The same applies to the Bass-Gura formula [4], which is not shown here.

In this paper, we consider a generalization of the Ackermann formula (7) for the case of a mathematical model of an EPS in the form of a dynamic system with many inputs (Multi Inputs Multi Outputs System – MIMO). This task is relevant, since there are no explicit calculation formulas for synthesizing the feedback law for MIMO systems similar to the Ackermann formula. Some exceptions are [5, 6]. In [5] to solve the problem, the Sylvester matrix equation and the associated inversion of the matrix polynomial (6), and in [6] is a special form of the control law, which significantly limits the area of its use.

In what follows, all transformations, taking into account duality, turn out to be valid for the problem of synthesis of the state observer.

Let a fully controlled MIMO system be given, which is understood as mathematical model of EPS,

$$\sigma \mathbf{x} = \boldsymbol{A}\mathbf{x} + \boldsymbol{B}\mathbf{u} \tag{8}$$

with state and control matrices of multiple dimensions
$\boldsymbol{A} \in \mathbb{R}^{k\cdot m}, \quad \boldsymbol{B} \in \mathbb{R}^{m}, \quad k, m \in \mathbb{N}$,
where the matrix

$$\Omega(\mathbf{A}, \mathbf{B}) = [\boldsymbol{B} \quad \boldsymbol{AB} \quad \dots \quad \boldsymbol{A}^{k-1}\boldsymbol{B}], \tag{9}$$

made up of the first k block columns of the Kalman controllability matrix is reversible. It is required to determine the matrices of regulators (observers) such that the state matrix "object - regulator" ("object - observer") $\boldsymbol{A} - \boldsymbol{BK}$ has given eigenvalues as $\boldsymbol{\Phi}^*$. At the same time, in the future we will compare the sets of solutions to this problem obtained by various analytical methods: using multilevel decomposition [7] and on the basis of a non-degenerate similarity transformation.

1. Solution using multilevel decomposition

Let us perform the following similarity transformation [8]:

$$\widetilde{\boldsymbol{A}} = \Omega^{-1}\boldsymbol{A}\Omega = \left[\begin{bmatrix}\mathbf{0}_{m\times(n-m)} \\ \boldsymbol{I}_{n-m}\end{bmatrix} \quad \Omega^{-1}\boldsymbol{A}^k\boldsymbol{B}\right],$$

$$\widetilde{\boldsymbol{B}} = \Omega^{-1}\boldsymbol{B} = \begin{bmatrix}\boldsymbol{I}_m \\ \mathbf{0}_{(n-m)\times m}\end{bmatrix}. \tag{10}$$

For the resulting pair of matrices, we synthesize the state controller with the matrix $\widetilde{\boldsymbol{K}}$.

In this case, when the condition(9), the number of decomposition levels of the pair of matrices $\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}}$, (including the zero level) is k [7]. The levels do not degenerate (the rank of the control matrices at each of the levels is m). Decomposition is performed according to the formulas [7]

$\widetilde{\boldsymbol{A}}_0 = \widetilde{\boldsymbol{A}}, \quad \widetilde{\boldsymbol{B}}_0 = \widetilde{\boldsymbol{B}},$

$\widetilde{\boldsymbol{A}}_{i+1} = \widetilde{\boldsymbol{B}}_i^{\perp L}\widetilde{\boldsymbol{A}}_i\widetilde{\boldsymbol{B}}_i^{\perp L}, \quad \widetilde{\boldsymbol{B}}_{i+1} = \widetilde{\boldsymbol{B}}_i^{\perp L}\widetilde{\boldsymbol{A}}_i\widetilde{\boldsymbol{B}}_i,$

$i = 0, 1, \dots, k-2,$

using semi-orthogonal left zero divisors

$\widetilde{\boldsymbol{B}}_i^{\perp L} = [\mathbf{0}_{(n-(i+1)m)\times m} \quad \boldsymbol{I}_{n-(i+1)m}], \quad i = 0, 1, \dots, k-2.$

The state matrices by decomposition levels take the form

$$\widetilde{\boldsymbol{A}}_i = \begin{bmatrix}\mathbf{0}_{m\times(n-(i+1)m)} & \widetilde{\boldsymbol{A}}_{(i+1,k)} \\ \boldsymbol{I}_{n-(i+1)m} & \widetilde{\boldsymbol{A}}_{(i+2:k,k)}\end{bmatrix} =$$

$$= \begin{bmatrix}\mathbf{0}_{m\times m} & \underbrace{\begin{bmatrix}\mathbf{0}_{m\times m} & \dots & \mathbf{0}_{m\times m} & \widetilde{\boldsymbol{A}}_{(i+1,k)}\end{bmatrix}}_{(k-1-i)\ \text{blocks}} \\ \widetilde{\boldsymbol{B}}_{i+1} & \widetilde{\boldsymbol{A}}_{i+1}\end{bmatrix},$$

$i = 0, 1, \dots, k-2,$

$\widetilde{\boldsymbol{A}}_{k-1} = \widetilde{\boldsymbol{A}}_{(k,k)}.$

The control matrices by decomposition levels have the form

$$\widetilde{\boldsymbol{B}}_i = \begin{bmatrix}\boldsymbol{I}_m \\ \mathbf{0}_{(n-(i+1)m)\times m}\end{bmatrix} = \begin{bmatrix}\widetilde{\boldsymbol{B}}_{i+1} \\ \mathbf{0}_{m\times m}\end{bmatrix}, \quad i = 0, 1, \dots, k-2,$$

$\widetilde{\boldsymbol{B}}_{k-1} = \boldsymbol{I}_m.$

Here and below, the designation corresponds to the block located at the intersection $\boldsymbol{A}_{l(i,j)}$ i-th block line and j-th a block column of a matrix $\boldsymbol{A}_l$ divided into blocks of the same size $m \times m$.

Assign matrices with given eigenvalues by decomposition levels

$\boldsymbol{\Phi}_0, \boldsymbol{\Phi}_1, \dots, \boldsymbol{\Phi}_{k-1} \in \mathbb{C}^{m\times m},$

such that

$$\bigcup_{i=0}^{k-1} eig\,\boldsymbol{\Phi}_i = \boldsymbol{\Phi}^*.$$

The controller matrix at the uppermost (k–1)-th level has the form

$\widetilde{\boldsymbol{K}}_{k-1} = \widetilde{\boldsymbol{B}}_{k-1}^{-1}\widetilde{\boldsymbol{A}}_{k-1} - \boldsymbol{\Phi}_{k-1}\widetilde{\boldsymbol{B}}_{k-1}^{-1} = \widetilde{\boldsymbol{A}}_{(k,k)} - \boldsymbol{\Phi}_{k-1}$

Next, we sequentially perform the calculations of the controller matrices at the levels. Using pseudoinverse and auxiliary matrices

$i = k-2, k-3, \dots, 1, 0,$

$\widetilde{\boldsymbol{B}}_i^{+} = \widetilde{\boldsymbol{B}}_i^{T}, \quad \widetilde{\boldsymbol{B}}_i^{-} = \widetilde{\boldsymbol{B}}_i^{+} + \widetilde{\boldsymbol{K}}_{i+1}\widetilde{\boldsymbol{B}}_i^{\perp L} = [\boldsymbol{I}_m \quad \widetilde{\boldsymbol{K}}_{i+1}]$

on i-th decomposition level, define the controller matrix

$\widetilde{\boldsymbol{K}}_i = \widetilde{\boldsymbol{B}}_i^{-}\widetilde{\boldsymbol{A}}_i - \boldsymbol{\Phi}_i\widetilde{\boldsymbol{B}}_i^{-}$

$= \begin{bmatrix}\widetilde{\boldsymbol{K}}_{i(1,1)} & \widetilde{\boldsymbol{K}}_{i(1,2)} & \dots & \widetilde{\boldsymbol{K}}_{i(1,k-i-1)} & \widetilde{\boldsymbol{K}}_{i(1,k-i)}\end{bmatrix},$

the blocks of which are written in terms of blocks of the controller matrix at the (i+1)-th level as follows:

$\widetilde{\boldsymbol{K}}_{i(1,1)} = \widetilde{\boldsymbol{K}}_{i+1(1,1)} - \boldsymbol{\Phi}_i$

$\widetilde{\boldsymbol{K}}_{i(1,2)} = \widetilde{\boldsymbol{K}}_{i+1(1,2)} - \boldsymbol{\Phi}_i\widetilde{\boldsymbol{K}}_{i+1(1,1)}$

…

$\widetilde{\boldsymbol{K}}_{i(1,k-i-1)} = \widetilde{\boldsymbol{K}}_{i+1(1,k-(i+1))} - \boldsymbol{\Phi}_i\widetilde{\boldsymbol{K}}_{i+1(1,k-(i+1)-1)}$

$$\widetilde{\boldsymbol{K}}_{i(1,k-i)} = \widetilde{\boldsymbol{A}}_{(i+1,k)} + \sum_{j=1}^{k-(i+1)}\left(\widetilde{\boldsymbol{K}}_{i+1(1,j)}\widetilde{\boldsymbol{A}}_{(i+1+j,k)}\right) - \boldsymbol{\Phi}_i\widetilde{\boldsymbol{K}}_{i+1(1,k-(i+1))}.$$

Let's rewrite the expressions for the blocks of the controller matrix

$\widetilde{\boldsymbol{K}}_{i(1,j)}$ $(j = 1, 2, \dots, k-i)(i = 0, 1, \dots, k-1)$ at the i-th level of decomposition $\widetilde{\boldsymbol{A}}$, $\widetilde{\boldsymbol{B}}$, in a recurrent form using the values of the known blocks of the same level, located to the left of the current block,

$$\widetilde{\boldsymbol{K}}_{i(1,j)} = \widetilde{\boldsymbol{S}}_{k-i,j} + \widetilde{\boldsymbol{A}}_{(k-(j-1),k)} + \sum_{r=1}^{j-1}\widetilde{\boldsymbol{K}}_{i(1,r)}\widetilde{\boldsymbol{A}}_{(k-(j-1)+r,k)}. \tag{11}$$

Here are the coefficients of the matrix polynomial $\boldsymbol{S}_{k-i,j}$

$$(\lambda \boldsymbol{I}_m - \boldsymbol{\Phi}_i)(\lambda \boldsymbol{I}_m - \boldsymbol{\Phi}_{i+1}) \dots (\lambda \boldsymbol{I}_m - \boldsymbol{\Phi}_{k-1}) = \\ = \lambda^{k-i}\boldsymbol{S}_{k-i,0} + \lambda^{k-i-1}\boldsymbol{S}_{k-i,1} + \lambda^{k-i-2}\boldsymbol{S}_{k-i,2} + \cdots \\ + \lambda \boldsymbol{S}_{k-i,k-i-1} + \boldsymbol{S}_{k-i,k-i}$$

Comparing the values of the blocks of the controller matrix at the lowest level with the values of the blocks in the lower k-th rows of matrices , we obtain the value of the controller matrix in explicit form

$\widetilde{\boldsymbol{K}}_{0(1,j)}\widetilde{\boldsymbol{K}} = \widetilde{\boldsymbol{K}}_0 i = 0\widetilde{\boldsymbol{A}}^0, \widetilde{\boldsymbol{A}}^1, \dots, \widetilde{\boldsymbol{A}}^k$

$$\widetilde{\boldsymbol{K}} = \begin{bmatrix}\widetilde{\boldsymbol{K}}_{0(1,1)} & \widetilde{\boldsymbol{K}}_{0(1,2)} & \dots & \widetilde{\boldsymbol{K}}_{0(1,k)}\end{bmatrix} = \\ \sum_{j=0}^{k}\left(\boldsymbol{S}_{k,j}\underbrace{[\boldsymbol{0}_{m\times m} \quad \dots \quad \boldsymbol{0}_{m\times m} \quad \boldsymbol{I}_m]}_{k \text{ blocks}}\widetilde{\boldsymbol{A}}^{k-j}\right). \tag{12}$$

Let us carry out the reverse transition by the similarity transformation (10)

$\boldsymbol{K} = \widetilde{\boldsymbol{K}}\Omega^{-1}$

and write the result

$$\boldsymbol{K} = \\ \sum_{j=0}^{k}\left(\boldsymbol{S}_{k,j}\underbrace{[\boldsymbol{0}_{m\times m} \quad \dots \quad \boldsymbol{0}_{m\times m} \quad \boldsymbol{I}_m]}_{k \text{ blocks}}\Omega^{-1}\boldsymbol{A}^{k-j}\right). \tag{13}$$

Recall that here are the coefficients of the matrix polynomial$\boldsymbol{S}_{k,j}$ ($j = 1, 2, \dots, k$)

$$(\lambda \boldsymbol{I}_m - \boldsymbol{\Phi}_0)(\lambda \boldsymbol{I}_m - \boldsymbol{\Phi}_1) \dots (\lambda \boldsymbol{I}_m - \boldsymbol{\Phi}_{k-1}) = \\ = \lambda^k \boldsymbol{S}_{k,0} + \lambda^{k-1}\boldsymbol{S}_{k,1} + \lambda^{k-2}\boldsymbol{S}_{k,2} + \cdots + \lambda \boldsymbol{S}_{k,k-1} + \boldsymbol{S}_{k,k}, \tag{14}$$

from which (in accordance with the theorem on the equality of spectra in the method of multilevel decomposition [7]) the characteristic polynomial of the matrix is formed $\boldsymbol{A} - \boldsymbol{BK}$

$$\left|\lambda^k \boldsymbol{S}_{k,0} + \lambda^{k-1}\boldsymbol{S}_{k,1} + \lambda^{k-2}\boldsymbol{S}_{k,2} + \cdots + \lambda \boldsymbol{S}_{k,k-1} + \boldsymbol{S}_{k,k}\right| \\ = |\lambda \boldsymbol{I}_m - \boldsymbol{\Phi}_0||\lambda \boldsymbol{I}_m - \boldsymbol{\Phi}_1| \dots |\lambda \boldsymbol{I}_m \\ - \boldsymbol{\Phi}_{k-1}|.$$

Formula (13) describes the set of equivalent solutions (regulator matrices) of the modal synthesis problem under consideration, and the parameterization of this set (the construction of the set of solutions)is carried out by all possible matrix polynomials (14).

2. Solution based on a nondegenerate similarity transformation

For a pair of matrices $(\boldsymbol{A}, \boldsymbol{B})$ perform a similarity transformation [8]. Note that in the general case the Yokoyama transformation [7], which removes restrictions on the ratio (multiplicity) of the dimensions of the state vectors and inputs used in this work. However, to simplify calculations, we will use the transformation(10).

As a result of the transformation (10) we obtain an equivalent pair of matrices $(\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}})$, for which we define a controller with matrix $\widetilde{\boldsymbol{K}}$. Next, we form the characteristic polynomial of the matrix $\widetilde{\boldsymbol{A}} - \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{K}}$:

$$|\lambda \boldsymbol{I}_n - (\widetilde{\boldsymbol{A}} - \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{K}})| \\ = \begin{vmatrix} \lambda \boldsymbol{I}_m + \widetilde{\boldsymbol{K}}_{(1,1)} & \widetilde{\boldsymbol{K}}_{(1,2)} & \widetilde{\boldsymbol{K}}_{(1,3)} & \widetilde{\boldsymbol{K}}_{(1,4)} & \dots & \widetilde{\boldsymbol{K}}_{1,(k-1)} & \widetilde{\boldsymbol{K}}_{(1,k)} - \widetilde{\boldsymbol{A}}_{(1,k)} \\ -\boldsymbol{I}_m & \lambda \boldsymbol{I}_m & \boldsymbol{0}_{m\times m} & \boldsymbol{0}_{m\times m} & \dots & \boldsymbol{0}_{m\times m} & -\widetilde{\boldsymbol{A}}_{(2,k)} \\ \boldsymbol{0}_{m\times m} & -\boldsymbol{I}_m & \lambda \boldsymbol{I}_m & \boldsymbol{0}_{m\times m} & \dots & \boldsymbol{0}_{m\times m} & -\widetilde{\boldsymbol{A}}_{(3,k)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \boldsymbol{0}_{m\times m} & \dots & \boldsymbol{0}_{m\times m} & -\boldsymbol{I}_m & \lambda \boldsymbol{I}_m & \boldsymbol{0}_{m\times m} & -\widetilde{\boldsymbol{A}}_{(k-2,k)} \\ \boldsymbol{0}_{m\times m} & \dots & \boldsymbol{0}_{m\times m} & \boldsymbol{0}_{m\times m} & -\boldsymbol{I}_m & \lambda \boldsymbol{I}_m & -\widetilde{\boldsymbol{A}}_{(k-1,k)} \\ \boldsymbol{0}_{m\times m} & \dots & \boldsymbol{0}_{m\times m} & \boldsymbol{0}_{m\times m} & \boldsymbol{0}_{m\times m} & -\boldsymbol{I}_m & \lambda \boldsymbol{I}_m - \widetilde{\boldsymbol{A}}_{(k,k)} \end{vmatrix}.$$

Here the determinant is divided by solid lines into blocks

$$\left|\lambda \boldsymbol{I}_n - (\widetilde{\boldsymbol{A}} - \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{K}})\right| = \begin{vmatrix}\boldsymbol{X}_{1,1} & \boldsymbol{X}_{1,2} \\ \boldsymbol{X}_{2,1} & \boldsymbol{X}_{2,2}\end{vmatrix}.$$

Since the block $\boldsymbol{X}_{2,1}$ is an invertible matrix, and , we write the equivalent characteristic equation$|\boldsymbol{X}_{2,1}| = (-1)^{n-m}$

$$\begin{vmatrix}\boldsymbol{X}_{1,1} & \boldsymbol{X}_{1,2} \\ \boldsymbol{X}_{2,1} & \boldsymbol{X}_{2,2}\end{vmatrix} = 0 \Leftrightarrow \left|\begin{bmatrix}\boldsymbol{X}_{1,1} & \boldsymbol{X}_{1,2} \\ \boldsymbol{X}_{2,1} & \boldsymbol{X}_{2,2}\end{bmatrix}\begin{bmatrix}-\boldsymbol{X}_{2,1}^{-1}\boldsymbol{X}_{2,2} & \boldsymbol{I}_{n-m} \\ \boldsymbol{I}_m & \boldsymbol{0}_{m\times(n-m)}\end{bmatrix}\right| = 0 \Leftrightarrow \\ \Leftrightarrow \begin{vmatrix}\boldsymbol{X}_{1,2} - \boldsymbol{X}_{1,1}\boldsymbol{X}_{2,1}^{-1}\boldsymbol{X}_{2,2} & \boldsymbol{X}_{1,1} \\ \boldsymbol{0}_{(n-m)\times m} & \boldsymbol{X}_{2,1}\end{vmatrix} = 0 \Leftrightarrow |\boldsymbol{X}_{2,1}||\boldsymbol{X}_{1,2} - \boldsymbol{X}_{1,1}\boldsymbol{X}_{2,1}^{-1}\boldsymbol{X}_{2,2}| = 0 \Leftrightarrow \\ \Leftrightarrow |\boldsymbol{X}_{1,2} - \boldsymbol{X}_{1,1}\boldsymbol{X}_{2,1}^{-1}\boldsymbol{X}_{2,2}| = 0. \tag{15}$$

Next, we define the inverse matrix

$$\boldsymbol{X}_{2,1}^{-1} \\ = -\begin{bmatrix}\boldsymbol{I}_m & \lambda \boldsymbol{I}_m & \lambda^2 \boldsymbol{I}_m & \lambda^3 \boldsymbol{I}_m & \cdots & \lambda^{k-2}\boldsymbol{I}_m \\ \boldsymbol{0}_{m\times m} & \boldsymbol{I}_m & \lambda \boldsymbol{I}_m & \lambda^2 \boldsymbol{I}_m & \cdots & \lambda^{k-3}\boldsymbol{I}_m \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{0}_{m\times m} & \cdots & \boldsymbol{0}_{m\times m} & \boldsymbol{I}_m & \lambda \boldsymbol{I}_m & \lambda^2 \boldsymbol{I}_m \\ \boldsymbol{0}_{m\times m} & \cdots & \boldsymbol{0}_{m\times m} & \boldsymbol{0}_{m\times m} & \boldsymbol{I}_m & \lambda \boldsymbol{I}_m \\ \boldsymbol{0}_{m\times m} & \cdots & \boldsymbol{0}_{m\times m} & \boldsymbol{0}_{m\times m} & \boldsymbol{0}_{m\times m} & \boldsymbol{I}_m\end{bmatrix}$$

and matrix polynomial

$$\boldsymbol{X}_{1,2} - \boldsymbol{X}_{1,1}\boldsymbol{X}_{2,1}^{-1}\boldsymbol{X}_{2,2} = \lambda^k \widetilde{\boldsymbol{S}}_{k,0} + \lambda^{k-1}\widetilde{\boldsymbol{S}}_{k,1} + \lambda^{k-2}\widetilde{\boldsymbol{S}}_{k,2} + \\ \cdots + \lambda \widetilde{\boldsymbol{S}}_{k,k-1}\phi + \widetilde{\boldsymbol{S}}_{k,k}, \tag{15}$$

where by grouping like terms in powers ϕ matrix coefficients are determined

$$\widetilde{\boldsymbol{S}}_{k,j} = \widetilde{\boldsymbol{K}}_{(1,j)} - \widetilde{\boldsymbol{A}}_{(k-(j-1),k)} - \sum_{r=1}^{j-1}\widetilde{\boldsymbol{K}}_{(1,r)}\widetilde{\boldsymbol{A}}_{(k-(j-1)+r,k)}.$$

From here one can express

$$\widetilde{\boldsymbol{K}}_{(1,j)} = \widetilde{\boldsymbol{S}}_{k,j} + \widetilde{\boldsymbol{A}}_{(k-(j-1),k)} + \sum_{r=1}^{j-1}\widetilde{\boldsymbol{K}}_{(1,r)}\widetilde{\boldsymbol{A}}_{k-(j-1)+r,k}.$$

The resulting formula is similar to the expression (11) for $i = 0$ up to coefficients , which are now replaced by . Therefore, by analogy with $\boldsymbol{S}_{k,j}\widetilde{\boldsymbol{S}}_{k,j}$(12) and (13) we get

$$\widetilde{\boldsymbol{K}} = \begin{bmatrix}\widetilde{\boldsymbol{K}}_{(1,1)} & \widetilde{\boldsymbol{K}}_{(1,2)} & \dots & \widetilde{\boldsymbol{K}}_{(1,k)}\end{bmatrix} = \\ \sum_{i=0}^{k}\left(\widetilde{\boldsymbol{S}}_{k,i}\underbrace{[\boldsymbol{0}_{m\times m} \quad \dots \quad \boldsymbol{0}_{m\times m} \quad \boldsymbol{I}_m]}_{k \text{ blocks}}\widetilde{\boldsymbol{A}}^{k-i}\right) \tag{16}$$

and

$$\boldsymbol{K} = \sum_{j=0}^{k}\left(\widetilde{\boldsymbol{S}}_{k,j}\underbrace{[\boldsymbol{0}_{m\times m} \quad \dots \quad \boldsymbol{0}_{m\times m} \quad \boldsymbol{I}_m]}_{k \text{ blocks}}\Omega^{-1}\boldsymbol{A}^{k-j}\right). \tag{17}$$

Expression (18) describes the set of equivalent solutions (regulator matrices) of the problem under consideration, the parameterization of which is carried out by all possible matrix polynomials (16). The main difference between this solution and (13) is that the matrix polynomial (16) does not have to be decomposed into matrix factors, as is done in (14). The main thing is that the roots of the characteristic equation (15)

$$\left|\lambda^k \boldsymbol{I}_m + \lambda^{k-1}\tilde{\boldsymbol{S}}_{k,1} + \lambda^{k-2}\tilde{\boldsymbol{S}}_{k,2} + \cdots + \lambda\tilde{\boldsymbol{S}}_{k,k-1} + \tilde{\boldsymbol{S}}_{k,k}\right| = 0$$

coincided with the corresponding eigenvalues of the matrices $\boldsymbol{\Phi}^*$.

3. Example

Consider an example of practical synthesis, when the task is to modify the previously obtained control law for synchronous generators so that the modes of electromechanical oscillations provided by the original control laws are preserved, while the modified laws would satisfy some additional requirements. Such requirements may include "enhanced" roughness with respect to disturbances (robustness), or, conversely, increased sensitivity to changes in controlled parameters in a given region or frequency band, etc. These requirements can be met by varying the given matrices $\bigcup_{i=0}^{k-1} eig\,\boldsymbol{\Phi}_i = \boldsymbol{\Phi}^*$. that the desired solution of the modal control problem under consideration, which belongs to the set of possible solutions (18), is not contained in the set (13), obtained using multilevel decomposition. In other words, consider the case when the matrix polynomial (16) does not decompose into matrix factors by analogy with the controller matrix (14).

Let the model be given EES, including synchronous generators and represented by the following linearized equations $N/2$:

$$\frac{d\delta_i}{dt} = s_i, \frac{ds_i}{dt} = -\frac{1}{J_i}(\Delta P_i + D_i \Delta s_i - \Delta u_i),\ i = 1,2,\ldots,k = N/2. \quad (19)$$

Here D_i is the damping coefficient, J_i is the moment of inertia, ΔP_i is the change in active power, determined using the equations of the electrical network

$$\begin{bmatrix}\Delta \boldsymbol{P}\\ \Delta \boldsymbol{Q}\end{bmatrix} = \begin{bmatrix}\boldsymbol{H} & \boldsymbol{M}\\ \boldsymbol{N} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\Delta\delta\\ \Delta U\end{bmatrix},$$

Where $\Delta\mathbf{P}$ is the active power increment vector in all system nodes, $\Delta\mathbf{Q}$ is the reactive power increment vector, $\Delta\mathbf{U}$ is the voltage amplitude increment vector, $\Delta\boldsymbol{\delta}$ is the voltage phase shift angle increment vector, $\Delta\mathbf{s}$ is the generator slip increment vector, $\Delta\mathbf{u}$ is the control vector (signals of PID-controllers). The elements of the Jacobi matrix are the corresponding partial derivatives

$$H_{ij} = \frac{\partial P_i}{\partial \delta_j}, M_{ij} = \frac{\partial P_i}{\partial U_j}, N_{ij} = \frac{\partial Q_i}{\partial \delta_j}, D_{ij} = \frac{\partial Q_i}{\partial U_j}.$$

Under the assumptions made, the matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ in the vector equation of the model EES take the form

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1\\ -\frac{\bar{H}_{11}}{T_{J1}} & -\frac{\bar{H}_{12}}{T_{J1}} & \cdots & -\frac{\bar{H}_{1,k}}{T_{J1}} & -\frac{\bar{D}_{11}}{T_{J1}} & 0 & \cdots & 0\\ -\frac{\bar{H}_{21}}{T_{J2}} & -\frac{\bar{H}_{22}}{T_{J2}} & \cdots & -\frac{\bar{H}_{2,k}}{T_{J2}} & 0 & -\frac{\bar{D}_{2}}{T_{J2}} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -\frac{\bar{H}_{k-1,1}}{T_{Jk}} & -\frac{\bar{H}_{k-1,2}}{T_{Jk}} & \cdots & -\frac{\bar{H}_{k,k}}{T_{Jk}} & 0 & 0 & \cdots & -\frac{\bar{D}_{k}}{T_{Jk}} \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\\ b_1 & 0 & \cdots & 0\\ 0 & b_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & b_k \end{bmatrix}.$$

Here, as you can see, the dimension of inputs is a multiple of the dimension of states, and phase angles and slips are used as outputs.

Let us consider the case of specifying the parameters of the control law in the PID-controllers, which provide the solution of the problem formulated earlier.

Let and be given an arbitrary non-zero real number $m = 2, k = 2, n = k \cdot m = 4\kappa.$

Let also the matrices from (14) have the form $\boldsymbol{\Phi}^* = \{\phi^*, \ldots, \phi^*\}$

$$\tilde{\boldsymbol{S}}_{2,1} = \begin{bmatrix} -2\phi^* & 0\\ 0 & -2\phi^* \end{bmatrix},\quad \tilde{\boldsymbol{S}}_{2,2} = \begin{bmatrix} \phi^*\phi^* & \kappa\\ 0 & \phi^*\phi^* \end{bmatrix}.$$

Characteristic equation

$$\left|\lambda^2 \boldsymbol{I}_m + \lambda\tilde{\boldsymbol{S}}_{2,1} + \tilde{\boldsymbol{S}}_{2,2}\right| = \begin{vmatrix} (\lambda - \phi^*)^2 & \kappa\\ 0 & (\lambda - \phi^*)^2 \end{vmatrix} = 0$$

has roots coinciding with given eigenvalues (given modes of electromechanical oscillations), but the system of equations

$$\begin{cases} \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 = -\tilde{\boldsymbol{S}}_{2,1}\\ \boldsymbol{\Phi}_0\boldsymbol{\Phi}_1 = \tilde{\boldsymbol{S}}_{2,2} \end{cases} \quad (18)$$

turns out to be unsolvable with respect to the matrices $\boldsymbol{\Phi}_0, \boldsymbol{\Phi}_1 \in \mathbb{C}$.

Let further and $\boldsymbol{\Phi}^* = \{\phi_x^*, \phi_x^*, \phi_y^*, \phi_y^*\}$

$$\boldsymbol{S}_{2,1} = \begin{bmatrix} -2\phi_x^* & \kappa\\ 0 & -2\phi_y^* \end{bmatrix},\quad \boldsymbol{S}_{2,2} = \begin{bmatrix} \phi_x^*\phi_x^* & 0\\ 0 & \phi_y^*\phi_y^* \end{bmatrix}.$$

Characteristic equation

$$\left|\lambda^2 \boldsymbol{I}_m + \lambda\boldsymbol{S}_{2,1} + \boldsymbol{S}_{2,2}\right| = \begin{vmatrix} (\lambda - \phi_x^*)^2 & \kappa\phi\\ 0 & (\lambda - \phi_y^*)^2 \end{vmatrix} = 0$$

has roots coinciding with the given eigenvalues, and the system of equations

$$\begin{cases} \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 = -\boldsymbol{S}_{2,1}\\ \boldsymbol{\Phi}_0\boldsymbol{\Phi}_1 = \boldsymbol{S}_{2,2} \end{cases} \quad (19)$$

is solvable with respect to matrices . One of the two possible solutions to system (21) has the following simple form:$\boldsymbol{\Phi}_0, \boldsymbol{\Phi}_1 \in \mathbb{C}$

$$\boldsymbol{\Phi}_0 = \begin{bmatrix} \phi_x^* & -\frac{\kappa\phi_y^*}{\phi_x^* - \phi_y^*}\\ 0 & \phi_y^* \end{bmatrix},\quad \boldsymbol{\Phi}_1 = \begin{bmatrix} \phi_x^* & \frac{\kappa\phi_x^*}{\phi_x^* - \phi_y^*}\\ 0 & \phi_y^* \end{bmatrix}, \quad (20)$$

$$eig\,\boldsymbol{\Phi}_0 = \{\phi_x^*, \phi_y^*\},\qquad eig\,\boldsymbol{\Phi}_1 = \{\phi_x^*, \phi_y^*\},$$

where, as can be seen, the eigenvalues are located on the main diagonal of the corresponding matrices. Triangular matrices (22) have the following features:

– firstly, to exclude division by zero in the expressions of off-diagonal elements in matrices (22), it is necessary to prevent the coincidence of the given eigenvalues (multiplicity of oscillation modes);

– secondly, the off-diagonal elements of matrices (22) contain the previously mentioned "adjusting" parameterκ, which makes it possible to provide the previously mentioned "enhanced" roughness with respect to disturbances, or increased sensitivity to changes in controlled parameters in a given region or frequency band (while maintaining the given oscillation modes).

4. Conclusion

For the EES, represented by a linear stationary dynamic system with many inputs and many outputs (MIMO-system), compact highly efficient analytical formulas for calculating the coefficients of the feedback law are obtained when solving the problem of modal control (given placement of eigenvalues) by the full state vector. The resulting analytical formulas are generalizations to MIMO-systems of the well-known Ackermann formula applied to systems with one input and many outputs (SIMO-systems). The transformations are based on a non-degenerate similarity transformation and the algorithm of the multi-step decomposition method. The obtained formulas are applicable to dynamic systems, in which the dimension of the state space is a multiple of the dimension of the system inputs. This limitation is removed by using the Yokoyama transform. The resulting formulas differ in terms of the parametrization of the set of equivalent laws. It is shown that there are cases when the set of equivalent control laws obtained using an analytical formula based on a non-degenerate similarity transformation is wider than the set of control laws formed on the basis of the multi-step decomposition method. The synthesis example demonstrated the possibility of changing the previously obtained control law for synchronous generators in order to preserve the previous oscillation modes and satisfy some additional requirements for robustness and sensitivity. These requirements are provided by varying the parameters of special matrices. The implementation of the obtained laws involves the use of synchronized vector measurements $\boldsymbol{\Phi}^{*}$.

References

1. T. Kailath *Linear Systems*. Prentice Hall. Englewood Cliffs. NJ. 1980.
2. R. Dorf, R. Bishop *Modern control systems*. Moscow: Basic Knowledge Laboratory, 2002.
3. M.G. Gadzhiev, M.Sh. Misrikhanov, V.N. Ryabchenko, Yu.V. Sharov *Matrix methods of analysis and control of transient processes in electric power systems*. Moscow: MPEI Publishing House, 2019.
4. B. Kuo *Theory and design of digital control systems*. M.: Mashinostroenie, 1986.
5. K. Nordström, H. Norlander *On the multi input pole placement control problem* // Proc. 36 IEEE Conf. decision and control. 1997. V. 5. P. 4288 - 4293.
6. N.E. Zubov, E.A. Vorobieva, E.A. Mikrin, M.Sh. Misrikhanov, V.N. Ryabchenko *Synthesis of stabilizing control of a spacecraft based on the generalized Ackermann formula* // Izvestiya RAN, Theory and Control Systems, 2011, **No. 1**, pp. 96 – 106.
7. N.E. Zubov, E.A. Mikrin, V.N. Ryabchenko *Matrix methods in the theory and practice of aircraft automatic control systems*. M.: Publishing House of Moscow State Technical University named after N.E. Bauman, 2016.
8. N.E. Zubov, E.A. Mikrin, V.N. Ryabchenko *Synthesis of astatic control of a linear system based on the generalized Ackermann formula* // Bulletin of the Moscow State Technical University. N.E. Bauman. Ser. Instrumentation, 2017, **No. 1 (112)**, pp. 67 – 74.